Graduate School of Science
Hiroshima University
Special Selection for International Students

(Residing outside Japan)
[Admission in October 2017]

M A T H	Math.

ID Number	M

November 10th, 2016 10:00 – 12:00

NOTES

1. The following are being distributed.

English Question Sheets (including the cover)	4 sheets
Japanese Question Sheets (including the cover)	4 sheets
Answer Sheets	3 sheets
Draft Sheet	1 sheet

2. You will be given three questions.
3. Use one Answer Sheet for each question. Write the Question Number on each Answer Sheet. You may use the back side of the Answer Sheet if necessary.
4. Write your ID number on the cover of the Question Sheets, all the Answer Sheets and the Draft Sheet.
5. Hand in all the Answer Sheets and the Draft Sheet after the examination.

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Answer all the questions [1], [2] and [3].

[**1**] Let $M_2(\mathbb{C})$ be the set of all 2×2 complex matrices. When $A \in M_2(\mathbb{C})$, we define

$$\begin{cases} V_A &= \{X \in M_2(\mathbb{C}) \mid AX = X\}, \\ W_A &= \{X \in M_2(\mathbb{C}) \mid XA = X\}. \end{cases}$$

Define $B \in M_2(\mathbb{C})$ by $B = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$. Answer the following questions.

(1) For any $A \in M_2(\mathbb{C})$, prove that V_A is a linear subspace of $M_2(\mathbb{C})$ over $\mathbb{C}$.

(2) Find the eigenvalues and the eigenvectors of B.

(3) Find $\dim_{\mathbb{C}} V_B$.

(4) Find a basis of W_B as a vector space over $\mathbb{C}$.

(5) Prove that $\dim_{\mathbb{C}} V_A = \dim_{\mathbb{C}} W_A$ for any $A \in M_2(\mathbb{C})$.

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[**2**] Define the function f by $f(x) = \begin{cases} e^{-\frac{1}{x}} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$.

Answer the following questions.

(1) Find the derivative of $f(x)$ for $x > 0$.

(2) When $n > 1$, show that the n-th derivative of $f(x)$ for $x > 0$ is of the form

$$\frac{p_n(x)}{x^{2n}} e^{-\frac{1}{x}},$$

where $p_n(x)$ stands for a polynomial in x of degree at most $n - 1$.

(3) Show that $f(x)$ is a C^∞-function on $\mathbb{R}$.

(4) Show that $f(x)$ CANNOT be represented as a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

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[**3**] Answer the following questions.

(1) When X is a set, what does it mean that d is a distance on X? Give the definition.

(2) Let (X, d) be a distance space, i.e., the pair of a set X and a distance d on X. We say that a subset $U \subset X$ is an open subset if for any $x \in U$, there exists a positive number $\varepsilon > 0$ such that if $y \in X$ satisfies $d(x, y) < \varepsilon$, then $y \in U$ holds. Prove that this definition of open subsets determines a topology on X.

(3) Let (X, d_X) and (Y, d_Y) be distance spaces. When $f : X \to Y$ is a map, prove that the following 3 conditions are equivalent:

(a) For any $x \in X$ and for any positive number $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that if $y \in X$ satisfies $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$ holds.

(b) When $U \subset Y$ is an open subset, then $f^{-1}(U) \subset X$ is also an open subset.

(c) When a sequence $x_1, x_2, \ldots$ in X converges to $x_\infty \in X$, then the sequence $f(x_1), f(x_2), \ldots$ in Y converges to $f(x_\infty)$.