Graduate School of Science
Hiroshima University
Qualifying Examination

Department of Mathematics	MATHEMATICS

ID Number	M

November 9th, 2017	9:00 – 11:00	(Hanoi)
November 9th, 2017	10:00 – 12:00	(Beijing)

NOTES

1. The following are being distributed.

Question Sheets (including the cover)	4 sheets
Answer Sheets	3 sheets
Draft Sheets	3 sheets

2. You will be given three questions.
3. Use one Answer Sheet for each question. Write the Question Number on each Answer Sheet. You may use the back side of the Answer Sheet if necessary.
4. Write your ID number on the cover of the Question Sheets, all the Answer Sheets and the Draft Sheets.
5. Hand in all the Answer Sheets and the Draft Sheets after the examination.

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Answer all the questions [1], [2], and [3].

[1] Answer the following questions (A) and (B).

(A) Let $\mathbb{R}^3$ be the 3-dimensional real vector space with the standard inner product. We consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

Let V be the subspace of $\mathbb{R}^3$ generated by v_1, v_2, and let W be the subspace of $\mathbb{R}^3$ generated by v_3, v_4. Answer the following questions.

(1) Show that v_1 and v_2 form a basis of V.

(2) Find an orthonormal basis of V.

(3) Find a basis of $V \cap W$.

(B) We put $i = \sqrt{-1} \in \mathbb{C}$. Consider the following Hermitian matrix:

$$H = \begin{bmatrix} -1 & 1+i \\ 1-i & 0 \end{bmatrix}.$$

Find a unitary matrix U such that $D = {}^t\overline{U}\, H\, U$ is a diagonal matrix, where $\overline{U}$ is the matrix obtained from U by taking the complex conjugate of each component of U, and ${}^t\overline{U}$ is the transpose of $\overline{U}$.

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[**2**] Answer all the following questions:

(1) Let a be a positive real number. Show that the improper integral

$$I(a) = \int_0^\infty \exp(-x^2/a^2)dx$$

converges, and calculate $I(a)/I(1)$.

(2) Prove or disprove the following statement: The function $f(x) = 1/x$ defined on the space $\mathbb{R}_{>0}$ of positive real numbers is uniformly continuous.

(3) Find the maximum value of the function

$$f(x,y) = \frac{1 + x^2 + xy + y^2}{(1 + x^2 + y^2)^2}$$

defined on $\mathbb{R}^2$.

(4) Evaluate the following integral:

$$\iint_D xy^2 dxdy, \qquad \text{where} \quad D = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 4 \}.$$

(5) Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

diverges.

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[**3**] Answer the following questions (A) and (B).

(A) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x & (x \in \mathbb{Q}) \\ -x & (x \notin \mathbb{Q}). \end{cases}$$

(1) Show that $f(x)$ is continuous at $x = 0$.

(2) Determine whether $f(x)$ is differentiable at $x = 0$ or not.

(B) Let $\{ a_n \mid n \in \mathbb{N} \}$ be a sequence of real numbers, where $\mathbb{N}$ is the set of positive integers. Let γ be a real number.

(1) Suppose that $\{a_n\}$ is split into two subsequences $\{b_n\}$, $\{c_n\}$; that is, we have subsequences $b_n = a_{i_n}$ $(i_1 < i_2 < \cdots)$ and $c_n = a_{j_n}$ $(j_1 < j_2 < \cdots)$ such that, if we put $I = \{i_1, i_2, \ldots\}$ and $J = \{j_1, j_2, \ldots\}$, then we have $I \cap J = \emptyset$ and $I \cup J = \mathbb{N}$.
Assume that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \gamma$. Show that $\lim_{n\to\infty} a_n = \gamma$ holds.

(2) Suppose that $\{a_n\}$ is split into infinite number of subsequences $\{b_n^{(1)}\}$, $\{b_n^{(2)}\}$, ... Determine whether the following is true in general:
If $\lim_{n\to\infty} b_n^{(s)} = \gamma$ holds for all s, then $\lim_{n\to\infty} a_n = \gamma$.